Superfluidity & Bogoliubov Theory: Rigorous Results

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Superfluidity

- matter behaves like a fluid with zero viscosity
- very low temperature
- discovered in 1937 for liquid Helium
- in trapped Bose-Einstein condensates, neutron stars,...

▶ A microscopic effect
- macroscopic manifestation of quantum mechanics
- essentially for bosons (Helium-4),
  more subtle for fermions (Helium-3)

▶ Related to Bose-Einstein condensation (?)
- discovery of quantized vortex lines (50s) and rings (60s)
- Gross & Pitaevskii in 1961 for liquid Helium
- only about 10 % of the particles are condensed in superfluid Helium

Top: lack of viscosity in superfluid Helium
Middle: The core of a trapped 2D cold Bose gas is superfluid (Dalibard et al, Nature Physics, 2012)
Bottom: numerical simulation of vortices in a BEC (GPE-lab, Antoine & Duboscq)
Microscopic origin of superfluidity

- Bogoliubov ('47), Feynman ('55) & Landau ('62): positive speed of sound in the gas, due to interactions between particles
- seen in excitation spectrum

\[
e(k) = |k|^2
\]

\[
e(k) \approx \sqrt{|k|^4 + 2\tilde{w}(k)|k|^2}
\]

(Bogoliubov assuming full BEC)

Theory

Bose-Einstein condensation

- (almost) all the particles in the gas behave the same
- their common wavefunction \( u \) solves the nonlinear Gross-Pitaevskii equation

\[
\left(|\nabla + iA(x)|^2 + V(x) + w * |u|^2\right)u = \begin{cases} 
i \partial_t u \\
\epsilon u \end{cases}
\]

Bogoliubov

- fluctuations of the condensate described by a linear equation

\[
\mathbb{H}_u \Phi = \begin{cases} 
i \partial_t \Phi \\
\lambda \Phi \end{cases}
\]

\( \mathbb{H}_u = \) Bogoliubov Hamiltonian

= (second) quantization of the Hessian of the GP energy at \( u \)

= has spectrum with the finite speed of sound when \( u=\)minimizer

Goal:

- prove this in appropriate regimes
- semi-classical theory in infinite dimension, with an effective semi-classical parameter
Regimes

▶ Two typical physical systems

- **Confined gas:** external potential $V(x) \to \infty$
  Studied a lot since the end of the 90s

- **Infinite gas:** $V \equiv 0$, infinitely many particles
  Very poorly understood mathematically

▶ Dilute regime: *rare collisions of order 1*

- low density $\rho \to 0$
- $w \sim 4\pi a\delta$
- very relevant physically
- confined gas: BEC proved by Lieb-Seiringer-Yngvason ’00s,... Bogoliubov open

▶ Mean-field regime: *many small collisions*

- high density $\rho \to \infty$, small interaction $\sim 1/\rho$
- good setting for the law of large numbers
- a bit less relevant physically
- confined gas: many works on BEC, Bogoliubov only understood very recently
- this talk

Many-particle mean-field Hamiltonian

- $N$ (spinless) bosons in $\mathbb{R}^d$ with $N \to \infty$
- A external magnetic potential or Coriolis force, $V$ external potential (e.g. lasers)
- two-body interaction $\lambda w$, with $\lambda \to 0$

$H_N = \sum_{j=1}^{N} |\nabla_{x_j} + iA(x_j)|^2 + V(x_j) + \lambda \sum_{1 \leq j < k \leq N} w(x_j - x_k), \quad \lambda \sim \frac{1}{N}$

acting on $L^2_s(\mathbb{R}^d)^N = \{ \Psi(x_1, \ldots, x_N) = \Psi(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) \in L^2, \quad \forall \sigma \in S_N \}$

Assumptions for this talk:
- $h = |\nabla + iA|^2 + V$ is bounded-below and has a compact resolvent on $L^2(\mathbb{R}^d)$
- $w$ is $h$–form-bounded with relative bound $< 1$. Can be attractive or repulsive or both

$\lambda_\ell(H_N) := \ell$th eigenvalue of $H_N$

Rmk. dilute regime corresponds to $w_N = N^{3\beta} w(N^\beta x)$ with $\beta = 1$ in $d = 3$, here $\beta = 0$. 

Mathieu LEWIN (CNRS / Paris-Dauphine) Superfluidity & Bogoliubov
Gross-Pitaevskii energy

If $\lambda = 0$, then the particles are all exactly iid

$$\Psi(x_1, \ldots, x_N) = u(x_1) \cdots u(x_N) = u^\otimes N(x_1, \ldots, x_N)$$

where $u \in L^2(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} |u(x)|^2 \, dx = 1$

$$\frac{\langle u^\otimes N, H_N u^\otimes N \rangle}{N} = \int_{\mathbb{R}^d} |\nabla u(x) + iA(x)u(x)|^2 \, dx + \int_{\mathbb{R}^d} V(x)|u(x)|^2 \, dx$$
$$+ \frac{(N - 1)\lambda}{2} \int_{\mathbb{R}^{2d}} w(x - y)|u(x)|^2|u(y)|^2 \, dx \, dy$$
$$= \mathcal{E}_{GP}(u), \quad \text{for } \lambda = \frac{1}{N - 1}$$

Minimizers:

$$\mathcal{E}_{GP}(u) := \inf_{\int_{\mathbb{R}^d} |u|^2 = 1} \mathcal{E}_{GP}(u)$$
Bose-Einstein condensation

Theorem (Derivation: ground state energy [M.L.-Nam-Rougerie ’14])

For every fixed $\ell \geq 1$ we have

$$\lim_{N \to \infty} \frac{\lambda_{\ell}(H_N)}{N} = e_{\text{GP}}.$$

Let $\Psi_N$ be such that $\langle \Psi_N, H_N \Psi_N \rangle = N e_{\text{GP}} + o(N)$. Then there exists a subsequence and a probability measure $\mu$, supported on $\mathcal{M} = \{\text{minimizers for } e_{\text{GP}}\}$, such that

$$\langle \Psi_{N_j}^{\otimes k}, A u^{\otimes k} \rangle_{\mathcal{M}} \to \int_{\mathcal{M}} \langle u^{\otimes k}, A u^{\otimes k} \rangle d\mu(u),$$

for every bounded operator $A$ on $L^2(\mathbb{R}^{dk})$ and every $k \geq 1$.

- Strong convergence of (quantum) marginals = density matrices
- Easy when $A = 0$ & $\hat{w} \geq 0$ is smooth
- Many works since the 80s (Fannes-Spohn-Verbeure ’80, Benguria-Lieb ’80, Lieb-Thirring-Yau ’84, Petz-Raggio-Verbeure ’89, Raggio-Werner ’89, Lieb-Seiringer ’00s,...)
- Our method, based on quantum de Finetti thms, also works for locally confined systems
- It can be used to simplify proof in dilute case (Nam-Rougerie-Seiringer ’15)

Describing fluctuations

Assumption

e_{\text{GP}} has a unique minimizer $u_0$ (up to a phase factor), which is non-degenerate.

Any symmetric function $\Psi$ of $N$ variables may be uniquely written in the form

$$\Psi = \varphi_0 u_0^\otimes N + \varphi_1 \otimes_s u_0^\otimes N - 1 + \varphi_2 \otimes_s u_0^\otimes N - 2 + \cdots + \varphi_N \otimes_s u_0^\otimes N - N + 1$$

with $\sum_{j=0}^N \|\varphi_j\|^2 = \|\Psi\|^2_{L^2}$. Here

$$f \otimes_s g(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \left( f(x_1, \ldots, x_k)g(x_{k+1}, \ldots, x_N) + \text{permutations} \right)$$

Natural to express the fluctuations using $\Phi = \varphi_0 \oplus \varphi_1 \oplus \cdots \oplus \varphi_N$.

In the limit $N \to \infty$, these live in the Fock space

$$\mathcal{F}_0 := \mathbb{C} \oplus \{u_0\}^\perp \oplus \bigoplus_{n \geq 2} \bigotimes_s \{u_0\}^\perp$$
Convergence of excitation spectrum

Theorem (Validity of Bogoliubov's theory [M.L.-Nam-Serfaty-Solovej '15])

We assume that $u_0$ is unique and non-degenerate, and that

$$
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} w(x - y)^2 |u_0(x)|^2 |u_0(y)|^2 \, dx \, dy < \infty.
$$

Let $H_0$ be the Bogoliubov Hamiltonian, that is, the second quantization of $\text{Hess} \, \mathcal{E}_{\text{GP}}(u_0)/2$ on $\mathcal{F}_0$. Then

- $\lambda_\ell(H_N) - N e_{\text{GP}} \to \lambda_\ell(H_0)$ for all $\ell \geq 1$;
- If $H_N \Psi_N = \lambda_\ell(H_N) \Psi_N$, then for a subsequence the corresponding fluctuations converge to a Bogoliubov eigenfunction: $\Phi_{N_j} \to \Phi = \varphi_0 \oplus \varphi_1 \oplus \cdots$ in $\mathcal{F}_0$ with $H_0 \Phi = \lambda_\ell(H_0) \Phi$. Equivalently,

$$
\left\| \Psi_{N_j} - \sum_{n=0}^{N_j} \varphi_n \otimes_s u_0 \otimes_{N_j-n} \right\| \to 0.
$$

- Generalizes Seiringer '11 and Grech-Seiringer '13
- Extension to isolated local minima by Nam-Seiringer '15
- For $\ell = 1$ $\varphi_{2j+1} \equiv 0 \ \forall j$, but usually $\varphi_{2j} \neq 0$ when $w \neq 0$
- $\Psi_N$ is not close to $(u_0)^{\otimes N}$ since usually $\Phi \neq \varphi_0$

Bogoliubov Hamiltonian

Hessian of GP energy

\[
\frac{1}{2} \text{Hess } \mathcal{E}_{\text{GP}}(u_0)(v, v) = \left\langle v, \left( |\nabla + iA|^2 + V + |u_0|^2 \ast w - \varepsilon_0 \right) v \right\rangle_{h_0} \\
+ \frac{1}{2} \int_{\mathbb{R}^2} w(x - y) \left( \overline{u_0(x)} u_0(y) v(x) v(y) + u_0(x) \overline{u_0(y)} v(x) v(y) + \text{c.c.} \right) \, dx \, dy \\
= \frac{1}{2} \left\langle \left( \begin{array}{c} v \\ \overline{v} \end{array} \right), \left( \begin{array}{cc} h_0 + K_1 & K_2^* \\ K_2 & h_0 + K_1 \end{array} \right) \left( \begin{array}{c} v \\ \overline{v} \end{array} \right) \right\rangle
\]

where \( K_1(x, y) = w(x - y) u_0(x) \overline{u_0(y)} \) and \( K_2(x, y) = w(x - y) u_0(x) u_0(y) \)

Bogoliubov \( \mathbb{H}_0 = \mathbb{H}_d + \mathbb{H}_p + (\mathbb{H}_p)^* \) where \( \mathbb{H}_d \) is diagonal and \( \mathbb{H}_p \) creates pair using the projection of \( K_2 \) on \( (\{u_0\}^\perp) \otimes s^2 \).

\( \mathbb{H}_0 \Phi = \lambda \Phi \) is an infinite system of linear equations:

\[
(\mathbb{H}_d)_n \varphi_n + (\mathbb{H}_p)_{n-2, n} \varphi_{n-2} + (\mathbb{H}_p)_{n+2, n} \varphi_{n+2} = \lambda \varphi_n
\]

\( \lambda \in \sigma(\mathbb{H}_0) \iff \left( \begin{array}{cc} h_0 + K_1 & K_2^* \\ K_2 & h_0 + K_1 \end{array} \right) \left( \begin{array}{c} u \\ v \end{array} \right) = \lambda \left( \begin{array}{c} u \\ -v \end{array} \right) \) (Bogoliubov-de Gennes)
A word on the dynamics

Theorem (Time-dependent Bogoliubov [M.L.-Nam-Schlein ’15])

Let \( u_0 \) with \( \int_{\mathbb{R}^d} |u_0|^2 = 1 \) and \( \langle u_0, hu_0 \rangle < \infty \). Let \( \Phi = (\varphi_n)_{n \geq 0} \in \mathcal{F}_{u_0} \) with \( \sum_{n \geq 0} \|\varphi_n\|^2 = 1 \) and \( \sum_{n \geq 0} \left\langle \varphi_n, \sum_{j=1}^n h_j \varphi_n \right\rangle < \infty \).

Then the solution of

\[
\begin{aligned}
\begin{cases}
i \dot{\Psi}_N &= H_N \Psi_N \\
\Psi_N(0) &= \sum_{n=0}^N \varphi_n \otimes_s u_0 \otimes_{N-n}
\end{cases}
\end{aligned}
\]

has converging fluctuations \( \Phi_N(t) \to \Phi(t) = \bigoplus_{n \geq 0} \varphi_n(t) \) for every \( t \), or equivalently,

\[
\left\| \Psi_N(t) - \sum_{n=0}^N \varphi_n(t) \otimes_s u(t) \otimes_{N-n} \right\| \to 0,
\]

where \[
\begin{aligned}
\begin{cases}
i \dot{u} &= (|\nabla + iA|^2 + V + w \ast |u|^2 - \varepsilon(t)) u \\
u(0) &= u_0
\end{cases}
\end{aligned}
\]

and \[
\begin{aligned}
\begin{cases}
i \dot{\Phi} &= \mathbb{H}(t) \Phi \\
\Phi(0) &= \Phi_0
\end{cases}
\end{aligned}
\]

with \( \mathbb{H}(t) \) the Bogoliubov Hamiltonian describing the excitations around \( u(t) \).

Hepp ’74, Ginibre-Velo ’79, Spohn ’80, Grillakis-Machedon-Margetis ’00s, Chen ’12, Deckert-Fröhlich-Pickl-Pizzo ’14, Benedikter-de Oliveira-Schlein ’14, ...

Cold Bose gases pose many interesting questions to mathematicians, which are also physically important.

All are still open for the infinite gas, even in the dilute and mean-field regimes.

Trapped gases are better understood since the beginning of the 00s.

**Mean-field microscopic model**

- simpler theory with weak interactions and high density
- physically justified in some special cases (stars, tunable interactions mediated through a cavity)
- full justification of Bose-Einstein Condensation & Bogoliubov excitation spectrum achieved only recently